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A novel proof of the existence of solutions for a new system of generalized mixed quasi-variational-like inclusions involving (A, η, m)-accretive operators

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Abstract In this paper, we introduce a new system of generalized mixed quasi-variational-like inclusions with (A, η, m)-accretive operators and relaxed cocoercive mappings. By using the fixed point theorem of Nadler, we prove the existence of solutions for this general system of generalized mixed quasi-variational-like inclusions and its special cases. The results in this paper unify, extend and improve some known results in the literature. The novel proof method is simpler than those iterative algorithm approach for proving the existence of solutions of all classes of system of set-valued variational inclusions in the literature.

Keywords System of generalized mixed quasi-variational-like inclusions · (A, η, m)-accretive operator · Relaxed cocoercive mapping · Fixed point theorem · Existence

1 Introduction

Recently, some new and interesting problems, which are called system of variational inclusions were introduced and studied. Fang and Huang [1], Verma [2], and Fang et al. [3], Fang and Huang [4], and Peng and Zhu [5] introduced and studied a system of variational inclusions involving H-monotone operators, A-monotone operators, (H, η)-monotone operators, H-accretive operators, and (H, η)-accretive operators, respectively. Peng [6] introduced and studied a new system of variational inclusions with (A, η, m)-accretive operators which contains systems of variational inclusions in [1–5] as special cases. By using the resolvent technique for the (A, η, m)-accretive operators, the author proved the existence and uniqueness of solution and the convergence of a new multi-step iterative algorithm for this system of variational inclusions in real q-uniformly smooth Banach spaces. Kazmi and Khan [7] introduce and study a new system of variational-like inclusions in real q-uniformly smooth Banach spaces.

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On the other hand, some classes of systems of set-valued variational inclusions were introduced and studied. Yan et al. [8] introduced and studied a system of set-valued variational inclusions involving H-monotone operators. Peng and Zhu [9–11] introduced some new system of generalized mixed quasi-variational inclusions with (H, η)-monotone operators. Peng [12] introduced and studied a new system of generalized mixed quasi-variational-like inclusions with (H, η)- accretive operators. Ding and Feng [13] introduced a new system of generalized mixed quasi-variational-like inclusions with (A, η)-accretive operators in real q-uniformly smooth Banach spaces which includes those mathematical models in [1–5,7,11,12]. By using the resolvent technique for the H-monotone operators, the (H, η)-monotone operators, the (H, η)-accretive operators, the (A, η, m)-accretive operators, respectively, the authors in [8–13] proved the the existence of solutions by proving the convergence of some iterative algorithms for those corresponding classes of systems of generalized mixed quasi-variational-like inclusions in Hilbert spaces or real q-uniformly smooth Banach spaces.

The purpose of this paper is twofold. It is easy to see that the system of variational inclusions in [6,9,10] is not special cases of those of mathematical models in [11–13]. Hence, we firstly will introduce a system of generalized mixed quasi-variational-like inclusions with (A, η, m)-accretive operators which is different from that in [13] and includes the mathematical models in [1–10] and the references therein. Secondly, it is easy to see that we can directly prove the existence of solutions for those classes of systems of variational inclusions in [1–7] without any iterative algorithms. So, we need to give an answer to the following question: Can we prove the existence of solutions for systems of set-valued variational inclusions such as those in [8–13]? In this paper, we will give a positive answer for this question. In other words, we can prove the existence of solutions for a new system of generalized mixed quasi-variational-like inclusions with (A, η, m)-accretive operators by a fixed point approach instead of those iterative algorithms methods in the literature. The results in this paper unify, extend and improve some known results in the literature.

2 Preliminaries

We suppose that E is a real Banach space with dual space, norm and the generalized dual pair denoted by E^*, $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively, 2^E is the family of all the nonempty subsets of E, $CB(E)$ is the families of all nonempty closed bounded subsets of E, and the generalized duality mapping $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \{f^* \in E^* : \langle x, f^*\rangle = \|f^*\| \cdot \|x\|, \|f^*\| = \|x\|^{q-1}\}, \quad \forall x \in E,$$

where $q > 1$ is a constant. In particular, J_2 is the usual normalized duality mapping. It is known that, in general, $J_q(x) = \|x\|^2 J_2(x)$, for all $x \neq 0$, and J_q is single-valued if E^* is strictly convex.

The modulus of smoothness of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(t) = sup\left\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| \leq 1, \|y\| \leq t\right\}.$$

A Banach space E is called uniformly smooth if

$$\lim_{t\to 0}\frac{\rho_E(t)}{t} = 0.$$

E is called q-uniformly smooth if there exists a constant $c > 0$, such that

$$\rho_E(t) \leq ct^q, q > 1.$$

Note that J_q is single-valued if E is uniformly smooth. Xu and Roach [14] proved the following result.

Lemma 2.1 *Let E be a real uniformly smooth Banach space. Then, E is q-uniformly smooth if and only if there exists a constants $c_q > 0$, such that for all $x, y \in E$,*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q.$$

We recall some definitions and results needed later.

Definition 2.1 [15] Let E be a real uniformly smooth Banach space, $T : E \longrightarrow E$ and $\eta : E \times E \longrightarrow E$ be two single-valued operators. T is said to be

(i) η-accretive if

$$\langle T(x) - T(y), J_q(\eta(x, y))\rangle \geq 0, \quad \forall x, y \in E;$$

(ii) strictly η-accretive if T is η-accretive and

$$\langle T(x) - T(y), J_q(\eta(x, y))\rangle = 0 \quad \text{if and only if } x = y;$$

(iii) r-strongly η-accretive if there exists a constant $r > 0$ such that

$$\langle T(x) - T(y), J_q(\eta(x, y))\rangle \geq r\|x - y\|^q, \quad \forall x, y \in E;$$

(iv) Lipschitz continuous if there exists a constant $s > 0$ such that

$$\|T(x) - T(y)\| \leq s\|x - y\|, \quad \forall x, y \in E.$$

Definition 2.2 [15] Let E be a real uniformly smooth Banach space, $T : E \longrightarrow E$ and $g : E \longrightarrow E$ be two single-valued operators. T is said to be

(i) (α, ξ)-relaxed cocoercive with respect to g if there exists constants $\alpha, \xi > 0$ such that

$$\langle T(x) - T(y), J_q(g(x) - g(y))\rangle \geq -\alpha\|T(x) - T(y)\|^q + \xi\|x - y\|^q, \quad \forall x, y \in E;$$

(ii) ξ-strongly accretive with respect to g if there exists constants $\alpha > 0$ such that

$$\langle T(x) - T(y), J_q(g(x) - g(y))\rangle \geq \xi\|x - y\|^q, \quad \forall x, y \in E;$$

Remark 2.1

(i) The (α, ξ)-relaxed cocoercivity with respect to I and the ξ-strongly accretivity with respect to I, respectively, are called the (α, ξ)-relaxed cocoercivity and the ξ-strongly accretivity, where I is the identity map on E.
(ii) If T is (α, ξ)-relaxed cocoercive (with respect to g), then T must be ξ-strongly accretivity (with respect to g). And the converse is not true in general.

Definition 2.3 [4,5,7] Let $\eta : E \times E \longrightarrow E$, $H : E \longrightarrow E$ be single-valued operators and $M : E \longrightarrow 2^E$ be a multi-valued operator. M is said to be

(i) accretive if

$$\langle u - v, J_q(x - y)\rangle \geq 0, \quad \forall x, y \in E, u \in M(x), v \in M(y);$$

(ii) η-accretive if

$$\langle u - v, J_q(\eta(x, y))\rangle \geq 0, \quad \forall x, y \in E, u \in M(x), v \in M(y);$$

(iii) strictly η-accretive if M is η-accretive and equality holds if and only if $x = y$;
(iv) r-strongly η-accretive if there exists a constant $r > 0$ such that

$$\langle u - v, J_q(\eta(x, y))\rangle \geq r\|x - y\|^q, \quad \forall x, y \in E, u \in M(x), v \in M(y);$$

(v) m-accretive if M is accretive and $(I + \rho M)(E) = E$ holds for all $\rho > 0$;
(vi) generalized η-accretive if M is η-accretive and $(I + \rho M)(E) = E$ holds for all $\rho > 0$;
(vii) H-accretive if M is accretive and $(H + \rho M)(E) = E$ holds for all $\rho > 0$;
(viii) (H, η)-accretive if M is η-accretive and $(H + \rho M)(E) = E$ holds for all $\rho > 0$.

Definition 2.4 [6,15] Let $\eta : E \times E \longrightarrow E$ be a single-valued operator and $M : E \longrightarrow 2^E$ be a multi-valued operator. M is said to be relaxed η-accretive with a constant m, if there exists a constant m such that

$$\langle u - v, J_q(\eta(x, y))\rangle \geq -m\|x - y\|^q, \quad \forall x, y \in E, \ u \in M(x), \ v \in M(y).$$

Definition 2.5 [6,15] Let $\eta : E \times E \longrightarrow E$, $A : E \longrightarrow E$ be single-valued operators and $M : E \longrightarrow 2^E$ be a multi-valued operator. M is said to be (A, η, m)-accretive if M is relaxed η-accretive with a constant m and $(A + \rho M)(E) = E$ holds for all $\rho > 0$.

Remark 2.2

(i) (A, η, m)-accretive operators is also called (A, η)-accretive operators by Lan et al. [15].
(ii) The definition of $(A, \eta, 0)$-accretive operators is that of (A, η)-accretive operators in [5,7,12] with $A = P$ or $A = H$. If $\eta(x, y) = x - y, \forall x, y \in E$, then the definition of $(A, \eta, 0)$-accretive operators becomes that of A-accretive operators in [4] with $A = H$. If $E = \mathcal{H}$ is a Hilbert space, the definition of (A, η, m)-accretive operator becomes that of (A, η, m)-monotone operators (i.e., (A, η)-monotone operators in [16]), the definition of H-accretive operators in [4] becomes that of H-monotone operators in [1,8], the definition of the (P, η)-accretive operators in [5,7] becomes that of (P, η)-monotone operators in [3,9–11] with $P = H$, if $\eta(x, y) = x - y, \forall x, y \in \mathcal{H}$, then the definition of (A, η, m)-monotone operators becomes that of A-monotone operators in [2].

Definition 2.6 [13] Let $\eta : E \times E \longrightarrow E$, $A : E \longrightarrow E$ be single-valued operators and $M : E \times E \longrightarrow 2^E$ be a multi-valued operator. M is said to be (A, η, m)-accretive in the first argument if M is relaxed η-accretive in the first argument with a constant m and for each $\omega \in E$, $(A + \rho M(., \omega))(E) = E$ holds for all $\rho > 0$.

Definition 2.7 [17] Let $\eta : E \times E \longrightarrow E$ be a single-valued operator, then $\eta(., .)$ is said to be τ-Lipschitz continuous, if there exists a constant $\tau > 0$ such that

$$\| \eta(u, v) \| \leq \tau\|u - v\|, \quad \forall u, v \in E.$$

Definition 2.8 [13] Let $\eta : E \times E \longrightarrow E$ be a single-valued operator, $A : E \longrightarrow E$ be a strictly η-accretive single-valued operator, and $M : E \times E \longrightarrow 2^E$ be (A, η, m)-accretive in the first argument, $m > 0$ and $\lambda > 0$ be constants. Then for each $\omega \in E$, the resolvent operator $R^{A,\eta}_{M(.,\omega),\lambda,m} : E \longrightarrow E$ associated with A, η, m, M, λ is defined by

$$R^{A,\eta}_{M(.,\omega),\lambda,m}(u) = (A + \lambda M(., \omega))^{-1}(u), \quad \forall u \in E.$$

Lemma 2.2 [13,15] *Let $\eta : E\times E \longrightarrow E$ be a Lipschitz continuous operator with a constant τ, $A : E \longrightarrow E$ be a strongly η-accretive operator with a constant γ and $M : E \times E \longrightarrow 2^E$ be an (A, η, m)-accretive operator in the first argument. Then for each $\omega \in E$, the resolvent operator $R^{A,\eta}_{M(.,\omega),\lambda,m} : E \longrightarrow E$ is Lipschitz continuous with a constant $\frac{\tau^{q-1}}{\gamma - m\lambda}$, i.e.,*

$$\|R^{\mathcal{A},\eta}_{M(.,\omega),\lambda,m}(x) - R^{\mathcal{A},\eta}_{M(.,\omega),\lambda,m}(y)\| \leq \frac{\tau^{q-1}}{\gamma - m\lambda}\|x - y\|, \quad \forall x, y \in E,$$

where $\lambda \in (0, \gamma/m)$ is a constant.

Remark 2.3 Under the conditions of Definition 2.8 and Lemma 2.2, the resolvent operator $R^{A,\eta}_{M(.,\omega),\lambda,m}$ is single-valued (see [13] and [15]).

Definition 2.9 [12,13] Let $E_1, E_2, \ldots, E_p$ be Banach spaces, $g_1 : E_1 \longrightarrow E_1$ and $N_1 : \prod_{j=1}^{p} E_j \longrightarrow E_1$ be two single-valued mappings.

(i) N_1 is said to be ξ-Lipschitz continuous in the first argument if there exists a constant $\xi > 0$ such that

$$\|N_1(x_1, x_2, \ldots, x_p) - N_1(y_1, x_2, \ldots, x_p)\| \leq \xi\|x_1 - y_1\|,$$
$$\forall x_1, y_1 \in E_1, x_j \in E_j \ (j = 2, 3, \ldots, p).$$

(ii) N_1 is said to be β-strongly accretive with respect to g_1 in the first argument if there exists a constant $\beta > 0$ such that

$$\langle N_1(x_1, x_2, \ldots, x_p) - N_1(y_1, x_2, \ldots, x_p), J_q(g_1(x_1) - g_1(y_1))\rangle \geq \beta\|x_1 - y_1\|^q,$$
$$\forall x_1, y_1 \in E_1, x_j \in E_j \ (j = 2, 3, \ldots, p).$$

(iii) N_1 is said to be (ζ, α)-relaxed cocoercive with respect to g_1 in the first argument if there exists a constant $\gamma > 0$ such that

$$\langle N_1(x_1, x_2, \ldots, x_p) - N_1(y_1, x_2, \ldots, x_p), J_q(g_1(x_1) - g_1(y_1))\rangle$$
$$\geq -\zeta\|N_1(x_1, x_2, \ldots, x_p) - N_1(y_1, x_2, \ldots, x_p)\|^2 + \alpha\|x_1 - y_1\|^2,$$
$$\forall x_1, y_1 \in \mathcal{H}_1, x_j \in \mathcal{H}_j \ (j = 2, 3, \ldots, p).$$

In a similar way, we can define the Lipschitz continuity and the strong accretivity (relaxed cocoercivity) of $N_i : \prod_{j=1}^{p} E_j \longrightarrow E_i$ with respect to $g_i : E_i \longrightarrow E_i$ in the i-th argument $(i = 2, 3, \ldots, p)$.

Let $\tilde{D}(\cdot, \cdot)$ denote the Hausdorff metric on $CB(E)$ defined by

$$\tilde{D}(A, B) = max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\}, \quad \forall A, B \in CB(E),$$

where $d(a, B) = \inf_{b\in B} \|a - b\|$, $d(A, b) = \inf_{a\in A} \|a - b\|$.

Definition 2.10 [18] Let E be a real uniformly smooth Banach space, and $V : E \longrightarrow CB(E)$ be a set-valued mapping. V is said to be ξ-$\widetilde{D}-$Lipschitz continuous if there exists a constant $\xi > 0$ such that

$$\widetilde{D}(V(u), V(v)) \leq \xi \|u - v\|, \quad \forall u, v \in E.$$

3 A system of generalized mixed quasi-variational-like inclusions and existence of solutions

In what follows, unless other specified, for each $i = 1, 2, \dots, p$, we always suppose that E_i is a real q-uniformly Banach space, $A_i, g_i : E_i \longrightarrow E_i, \eta_i : E_i \times E_i \longrightarrow E_i, F_i, G_i : \prod_{j=1}^{p} E_j \longrightarrow E_i$ are single-valued mappings, $T_{1i} : E_i \longrightarrow CB(E_i), T_{2i} : E_i \longrightarrow CB(E_i), \dots, T_{pi} : E_i \longrightarrow CB(E_i)$ are set-valued mappings and $M_i : E_i \times E_i \longrightarrow 2^{E_i}$ is an (A_i, η_i, m_i)-accretive operator in the first argument. Assume that $g_i(E_i) \cap M_i(., \omega_i) \neq \emptyset$ for each $\omega_i \in E_i$. We consider the following problem of finding $(x_1, x_2, \dots, x_p, y_{11}, y_{12}, \dots, y_{1p}, y_{21}, y_{22}, \dots, y_{2p}, \dots, y_{p1}, y_{p2}, \dots, y_{pp})$ such that for each $i = 1, 2, \dots, p$, $x_i \in E_i$, $y_{1i} \in T_{1i}(x_i)$, $y_{2i} \in T_{2i}(x_i), \dots, y_{pi} \in T_{pi}(x_i)$ and

$$0 \in F_i(x_1, x_2, \dots, x_p) + G_i(y_{i1}, y_{i2}, \dots, y_{ip}) + M_i(g_i(x_i), x_i). \tag{3.1}$$

The problem (3.1) is called a system of generalized mixed quasi-variational-like inclusions with (A, η, m)-accretive operators in real q-uniformly Banach spaces.

Below are some special cases of problem (3.1).

(i) If $M_i(., .) = M_i(.)$ is an (H_i, η_i)-accretive operators, then problem (3.1) becomes the following system of generalized mixed quasi-variational-like inclusions with (H, η)-accretive operators, which is to find $(x_1, x_2, \dots, x_p, y_{11}, y_{12}, \dots, y_{1p}, y_{21}, y_{22}, \dots, y_{2p}, \dots, y_{p1}, y_{p2}, \dots, y_{pp})$ such that for each $i = 1, 2, \dots, p, x_i \in E_i, y_{1i} \in T_{1i}(x_i)$, $y_{2i} \in T_{2i}(x_i)$,..., $y_{pi} \in T_{pi}(x_i)$ and

$$0 \in F_i(x_1, x_2, \dots, x_p) + G_i(y_{i1}, y_{i2}, \dots, y_{ip}) + M_i(g_i(x_i)). \tag{3.2}$$

This problem is different from problem (3.1) in [12] which is a special case of problem (3.1) in [13].

(ii) For $i, j = 1, 2, \dots, p$, if $T_{ij} \equiv I_j$ (the identity map on E_j), then problem (3.1) becomes the following problem of finding $(x_1, x_2, \dots, x_p) \in \prod_{i=1}^{p} E_i$ such that for each $i = 1, 2, \dots, p$,

$$0 \in F_i(x_1, x_2, \dots, x_p) + G_i(x_1, x_2, \dots, x_p) + M_i(g_i(x_i)). \tag{3.3}$$

This problem is called a system of variational inclusions with (A, η, m)-accretive operators introduced and studied by Peng [6].

(iii) For $i = 1, 2, \dots, p$, if $E_i = \mathcal{H}_i$ is a Hilbert spaces, and $M_i(., .) = M_i(.)$ is an (H_i, η_i)-monotone operators, then problem (3.1) becomes the following system of generalized mixed quasi-variational-like inclusions with (H, η)-monotone operators, which is to find $(x_1, x_2, \dots, x_p, y_{11}, y_{12}, \dots, y_{1p}, y_{21}, y_{22}, \dots, y_{2p}, \dots, y_{p1}, y_{p2}, \dots, y_{pp})$ such that for each $i = 1, 2, \dots, p$, $x_i \in \mathcal{H}_i$, $y_{1i} \in T_{1i}(x_i)$, $y_{2i} \in T_{2i}(x_i)$,..., $y_{pi} \in T_{pi}(x_i)$ and

$$0 \in F_i(x_1, x_2, \dots, x_p) + G_i(y_{i1}, y_{i2}, \dots, y_{ip}) + M_i(g_i(x_i)). \tag{3.4}$$

If $p = 3$, then problem (3.4) becomes the system of set-valued quasi-variational inclusions introduced and studied by Peng and Zhu [10].

If $p = 2$, then problem (3.4) reduces to the system of generalized mixed quasi-variational inclusions with (H, η)-monotone operators introduced and studied by Peng and Zhu [9], which is to find $(x_1, x_2, y_{11}, y_{12}, y_{21}, y_{22})$ such that $(x_1, x_2) \in E_1 \times E_2$, $y_{11} \in T_{11}(x_1)$, $y_{12} \in T_{12}(x_2)$, $y_{21} \in T_{21}(x_1)$, $y_{22} \in T_{22}(x_2)$ and

$$\begin{cases} 0 \in F_1(x_1, x_2) + G_1(y_{11}, y_{12}) + M_1(g_1(x_1)), \\ 0 \in F_2(x_1, x_2) + G_2(y_{21}, y_{22}) + M_2(g_2(x_2)). \end{cases} \tag{3.5}$$

It is easy to see that problem (3.5) is different from problem (3.2) in [11].

If $p = 2$, $G_1 = G_2 = 0$, M_i be P_i-η_i-accretive in the first argument (i=1, 2), then problem (3.1) reduces to the problem (4.1) in [7].

Since $T_{ji} : E_j \longrightarrow CB(E_i)$ has been replaced $T_{ji} : E_i \longrightarrow CB(E_i)$, problem (3.1) in this paper is different from the problem 3.1 in [13] which is not contained those mathematical models in [6,8–10] as special cases. Problem (3.1) can be regarded as a corrected version of the problem 3.1 in [13].

Lemma 3.1 *For $i = 1, 2, \ldots, p$, let $\eta_i : E_i \times E_i \longrightarrow E_i$ be a single-valued operator, $A_i : E_i \longrightarrow E_i$ be a strictly η_i-accretive operator and $M_i : E_i \longrightarrow 2^{E_i}$ be an (A_i, η_i, m_i)-accretive operator. Then $(x_1, x_2, \ldots, x_p, y_{11}, y_{12}, \ldots, y_{1p}, y_{21}, y_{22}, \ldots, y_{2p}, \ldots, y_{p1}, y_{p2}, \ldots, y_{pp})$ with $x_i \in E_i$, $y_{1i} \in T_{1i}(x_i)$, $y_{2i} \in T_{2i}(x_i), \ldots, y_{pi} \in T_{pi}(x_i)$ $(i = 1, 2, \ldots, p)$ is a solution of the problem* (3.1) *if and only if for each $i = 1, 2, \ldots, p$,*

$$g_i(x_i) = R^{A_i,\eta_i}_{M_i(.,x_i),\lambda_i,m_i}(A_i(g_i(x_i)) - \lambda_i F_i(x_1, x_2, \ldots, x_p) - \lambda_i G_i(y_{i1}, y_{i2}, \ldots, y_{ip})),$$

where $R^{A_i,\eta_i}_{M_i(.,x_i),\lambda_i,m_i} = (A_i + \lambda_i M_i(., x_i))^{-1}$, $\lambda_i > 0$ are constants.

Proof The fact directly follows from Definition 2.8. □

Now we present some existence results of solutions for problem (3.1) and its special cases without involving any iterative algorithms.

Theorem 3.1 *For $i = 1, 2, \ldots, p$, let $\eta_i : E_i \times E_i \longrightarrow E_i$ be τ_i-Lipshitz continuous, $A_i : E_i \longrightarrow E_i$ be γ_i-strongly η_i-accretive and δ_i-Lipschitz continuous, $g_i : E_i \longrightarrow E_i$ be (t_i, r_i)-relaxed cocoercive and s_i-Lipschitz continuous, $F_i : \prod_{k=1}^{p} E_k \longrightarrow E_i$ be (ζ_i, α_i)-relaxed cocoercive with respect to $\hat{g}_i$ in the i-th argument, β_{ij}-Lipschitz continuous in the j-th argument for $j = 1, \ldots, i-1, i, i+1, \ldots, p$, where $\hat{g}_i : E_i \longrightarrow E_i$ is defined by $\hat{g}_i(x_i) = A_i \circ g_i(x_i) = A_i(g_i(x_i)), \forall x_i \in E_i$, and $G_i : \prod_{k=1}^{p} E_k \longrightarrow E_i$ be ξ_{ij}-Lipschitz continuous in the j-th argument for $j = 1, 2, \ldots, p$, $M_i : E_i \times E_i \longrightarrow 2^{E_i}$ be (A_i, η_i, m_i)-accretive in the first argument, and the set-valued mappings $T_{1i} : E_i \longrightarrow CB(E_i)$, $T_{2i} : E_i \longrightarrow CB(E_i), \ldots, T_{pi} : E_i \longrightarrow CB(E_i)$ be l_{1i}-$\tilde{D}$-Lipschitz continuous, l_{2i}-$\tilde{D}$-Lipschitz,…, l_{pi}-$\tilde{D}$-Lipschitz continuous, respectively. In addition, if*

$$\|R^{A_i,\eta_i}_{M_i(.,x_i),\lambda_i,m_i}(z_i) - R^{A_i,\eta_i}_{M_i(.,\hat{x}_i),\lambda_i,m_i}(z_i)\| \le \mu_i \| x_i - \hat{x}_i \|, \forall x_i, \hat{x}_i, z_i \in E_i, \tag{3.6}$$

for all $i = 1, 2, \ldots, p$. If there exist constants λ_i such that $0 < \lambda_i < \frac{\gamma_i}{m_i}$ $(i = 1, 2, \ldots, p)$ and

$$\begin{cases} \sqrt[q]{1 - qr_1 + qt_1 s_1^q + c_q s_1{}^q} + \mu_1 \\ \quad + \frac{\tau_1^{q-1}}{\gamma_1 - \lambda_1 m_1} \left(\sqrt[q]{\delta_1{}^q s_1{}^q - q\lambda_1\alpha_1 + q\lambda_1\zeta_1\beta_{11}{}^q + c_q\lambda_1^q\beta_{11}{}^q} + \lambda_1\xi_{11}l_{11} \right) \\ \quad + \sum_{j=2}^{p} \frac{\lambda_j\tau_j^{q-1}}{\gamma_j - \lambda_j m_j} (\beta_{j1} + \xi_{j1}l_{j1}) < 1, \\ \sqrt[q]{1 - qr_2 + qt_2 s_2^q + c_q s_2{}^q} + \mu_2 \\ \quad + \frac{\tau_2^{q-1}}{\gamma_2 - \lambda_2 m_2} \left(\sqrt[q]{\delta_2{}^q s_2{}^q - q\lambda_2\alpha_2 + q\lambda_2\zeta_2\beta_{22}{}^q + c_q\lambda_2^q\beta_{22}{}^q} + \lambda_2\xi_{22}l_{22} \right) \\ \quad + \frac{\lambda_1\tau_1^{q-1}}{\gamma_1 - \lambda_1 m_1} (\beta_{12} + \xi_{12}l_{12}) + \sum_{j=3}^{p} (\beta_{j2} + \xi_{j2}l_{j2}) \frac{\lambda_j\tau_j^{q-1}}{\gamma_j - \lambda_j m_j} < 1, \\ \ldots \\ \sqrt[q]{1 - qr_p + qt_p s_p^q + c_q s_p{}^q} + \mu_p \\ \quad + \frac{\tau_p^{q-1}}{\gamma_p - \lambda_p m_p} \left(\sqrt[q]{\delta_p{}^q s_p{}^q - q\lambda_p\alpha_p + q\lambda_p\zeta_p\beta_{pp}{}^q + c_q\lambda_p^q\beta_{pp}{}^q} + \lambda_p\xi_{pp}l_{pp} \right) \\ \quad + \sum_{j=1}^{p-1} \frac{\lambda_j\tau_j^{q-1}}{\gamma_j - \lambda_j m_j} (\beta_{jp} + \xi_{jp}l_{jp}) < 1. \end{cases} \tag{3.7}$$

Then problem (3.1) admits a solution $(x_1, x_2, \ldots, x_p, y_{11}, y_{12}, \ldots, y_{1p}, y_{21}, y_{22}, \ldots, y_{2p}, \ldots, y_{p1}, y_{p2}, \ldots, y_{pp})$.

Proof For $i = 1, 2, \ldots, p$, define a mapping $\Gamma_{\lambda_i} : \Pi_{j=1}^p X_j \times \Pi_{j=1}^p X_j \to X_i$ by

$$\Gamma_{\lambda_i}(x_1, x_2, \ldots, x_p, y_{i1}, y_{i2}, \ldots, y_{ip}) = x_i - g_i(x_i) \\ + R_{M_i(.,x_i),\lambda_i,m_i}^{A_i,\eta_i}(A_i(g_i(x_i)) - \lambda_i F_i(x_1, x_2, \ldots, x_p) - \lambda_i G_i(y_{i1}, y_{i2}, \ldots, y_{ip})), \tag{3.8}$$

for all $(x_1, x_2, \ldots, x_p, y_{i1}, y_{i2}, \ldots, y_{ip}) \in \Pi_{j=1}^p X_j \times \Pi_{j=1}^p X_j$.

Now define the norm $\|\cdot\|_*$ on $\Pi_{j=1}^p X_j$ by

$$\|(x_1, x_2, \ldots, x_p)\|_* = \|x_1\| + \|x_2\| + \cdots + \|x_p\|, \quad \forall (x_1, x_2, \ldots, x_p) \in \Pi_{j=1}^p X_j$$

It is easy to see that $(\Pi_{j=1}^p X_j, \|\cdot\|_*)$ is a Banach space. By (3.8), for any $\lambda_i > 0 (i = 1, 2, \ldots, p)$, define $\Phi_{\lambda_1,\lambda_2,\ldots,\lambda_p} : \Pi_{j=1}^p X_j \to 2^{\Pi_{j=1}^p X_j}$ by

$$\Phi_{\lambda_1,\lambda_2,\ldots,\lambda_p}(x_1, x_2, \ldots, x_p) = \Big\{ \Big(\Gamma_{\lambda_1}(x_1, x_2, \ldots, x_p, y_{11}, y_{12}, \ldots, y_{1p}), \\ \Gamma_{\lambda_2}(x_1, x_2, \ldots, x_p, y_{21}, y_{22}, \ldots, y_{2p}), \ldots, \\ \Gamma_{\lambda_p}(x_1, x_2, \ldots, x_p, y_{p1}, y_{p2}, \ldots, y_{pp}) \Big) : \forall y_{1j} \in T_{1j}(x_j), \\ y_{2j} \in T_{2j}(x_j), \ldots, y_{pj} \in T_{pj}(x_j), j = 1, 2, \ldots, p. \Big\}$$

for all $(x_1, x_2, \ldots, x_p) \in \Pi_{j=1}^p X_j$.

For any $(x_1, x_2, \ldots, x_p) \in \Pi_{j=1}^p X_j$, since for $i, j = 1, 2, \ldots, p$, $T_{ij}(x_j) \in CB(X_j)$, $A_i, \eta_i, g_i, F_i, G_i$, $R_{M_i(.,x_i),\lambda_i,m_i}^{A_i,\eta_i}$ are continuous, we have $\Phi_{\lambda_1,\lambda_2,\ldots,\lambda_p}(x_1, x_2, \ldots, x_p) \in CB(\Pi_{j=1}^p X_j)$. Now we prove that $\Phi_{\lambda_1,\lambda_2,\ldots,\lambda_p}$ is a set-valued contractive mapping.

In fact, for any $(x_1, x_2, \ldots, x_p), (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_p) \in \Pi_{j=1}^{p} X_j$, and any $(a_1, a_2, \ldots, a_p) \in \Phi_{\lambda_1,\lambda_2,\ldots,\lambda_p}(x_1, x_2, \ldots, x_p)$, there exist $y_{1j} \in T_{1j}(x_j), y_{2j} \in T_{2j}(x_j), \ldots, y_{pj} \in T_{pj}(x_j)$ $(j = 1, 2, \ldots, p)$ such that for each $i = 1, 2, \ldots, p$,

$$\begin{aligned} a_i &= x_i - g_i(x_i) + R^{A_i,\eta_i}_{M_i(.,x_i),\lambda_i,m_i}(A_i(g_i(x_i)) - \lambda_i F_i(x_1, x_2, \ldots, x_p) \\ &\quad -\lambda_i G_i(y_{i1}, y_{i2}, \ldots, y_{ip})), \end{aligned}$$

Note that for $i, j = 1, 2, \ldots, p$, $T_{ij}(\hat{x}_j) \in CB(X_j)$; it follows from Nadler's result [17] that there exist $\hat{y}_{1j} \in T_{1j}(\hat{x}_j), \hat{y}_{2j} \in T_{2j}(\hat{x}_j), \ldots, \hat{y}_{pj} \in T_{pj}(\hat{x}_j)$ $(j = 1, 2, \ldots, p)$ such that for each $i, j = 1, 2, \ldots, p$,

$$\|y_{ij} - \hat{y}_{ij}\| \le \widetilde{D}(T_{ij}(x_j), T_{ij}(\hat{x}_j)). \tag{3.9}$$

For $i = 1, 2, \ldots, p$, setting

$$\begin{aligned} b_i &= \hat{x}_i - g_i(\hat{x}_i) + R^{A_i,\eta_i}_{M_i(.,\hat{x}_i),\lambda_i,m_i}(A_i(g_i(\hat{x}_i)) - \lambda_i F_i(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_p) \\ &\quad -\lambda_i G_i(\hat{y}_{i1}, \hat{y}_{i2}, \ldots, \hat{y}_{ip})). \end{aligned}$$

We have $(b_1, b_2, \ldots, b_p) \in \Phi_{\lambda_1,\lambda_2,\ldots,\lambda_p}(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_p)$.

For $i = 1, 2, \ldots, p$, let $\Omega_i = A_i(g_i(x_i)) - \lambda_i F_i(x_1, x_2, \ldots, x_p) - \lambda_i G_i(y_{i1}, y_{i2}, \ldots, y_{ip})$, $\hat{\Omega}_i = A_i(g_i(\hat{x}_i)) - \lambda_i F_i(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_p) - \lambda_i G_i(\hat{y}_{i1}, \hat{y}_{i2}, \ldots, \hat{y}_{ip})$. By Lemma 2.2 and (3.6) that

$$\begin{aligned} \|a_i - b_i\| &= \|x_i - g_i(x_i) + R^{A_i,\eta_i}_{M_i(.,x_i),\lambda_i,m_i}(A_i(g_i(x_i)) - \lambda_i F_i(x_1, x_2, \ldots, x_p) \\ &\quad -\lambda_i G_i(y_{i1}, y_{i2}, \ldots, y_{ip})) - [\hat{x}_i - g_i(\hat{x}_i) + R^{A_i,\eta_i}_{M_i(.,\hat{x}_i),\lambda_i,m_i}(A_i(g_i(\hat{x}_i)) \\ &\quad -\lambda_i F_i(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_p) - \lambda_i G_i(\hat{y}_{i1}, \hat{y}_{i2}, \ldots, \hat{y}_{ip}))]\| \\ &\le \|x_i - \hat{x}_i - (g_i(x_i) - g_i(\hat{x}_i))\| + \|R^{A_i,\eta_i}_{M_i(.,x_i),\lambda_i,m_i}(\Omega_i) - R^{A_i,\eta_i}_{M_i(.,\hat{x}_i),\lambda_i,m_i}(\hat{\Omega}_i)\| \\ &\le \|x_i - \hat{x}_i - (g_i(x_i) - g_i(\hat{x}_i))\| + \|R^{A_i,\eta_i}_{M_i(.,x_i),\lambda_i,m_i}(\Omega_i) - R^{A_i,\eta_i}_{M_i(.,x_i),\lambda_i,m_i}(\hat{\Omega}_i)\| \\ &\quad +\|R^{A_i,\eta_i}_{M_i(.,x_i),\lambda_i,m_i}(\hat{\Omega}_i) - R^{A_i,\eta_i}_{M_i(.,\hat{x}_i),\lambda_i,m_i}(\hat{\Omega}_i)\| \\ &\le \|x_i - \hat{x}_i - (g_i(x_i) - g_i(\hat{x}_i))\| + \frac{\tau_i^{q-1}}{\gamma_i - \lambda_i m_i}\|\Omega_i - \hat{\Omega}_i\| + \mu_i \parallel x_i - \hat{x}_i \parallel, \\ &\quad i = 1, 2, \ldots, p. \end{aligned} \tag{3.10}$$

Since $g_i : E_i \longrightarrow E_i$ is (t_i, r_i)-relaxed cocoercive and s_i-Lipschitz continuous, by Lemma 2.1 we have,

$$\begin{aligned} &\|x_i - \hat{x}_i - (g_i(x_i) - g_i(\hat{x}_i))\|^q \\ &\quad \le \|x_i - \hat{x}_i\|^q + c_q\|g_i(x_i) - g_i(\hat{x}_i)\|^q - q\langle g_i(x_i) - g_i(\hat{x}_i), J_q(x_i - \hat{x}_i)\rangle \\ &\quad \le (1 + c_q s_i^q + q t_i s_i^q - q r_i)\|x_i - \hat{x}_i\|^q \end{aligned} \tag{3.11}$$

And

$$
\begin{aligned}
&\|\Omega_i - \hat{\Omega}_i\| \\
&\quad = \|A_i(g_i(x_i)) - \lambda_i F_i(x_1, x_2, \ldots, x_p) - \lambda_i G_i(y_{i1}, y_{i2}, \ldots, y_{ip}) \\
&\qquad -[A_i(g_i(\hat{x}_i)) - \lambda_i F_i(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_p) - \lambda_i G_i(\hat{y}_{i1}, \hat{y}_{i2}, \ldots, \hat{y}_{ip})]\| \\
&\quad \le \|A_i(g_i(x_i)) - A_i(g_i(\hat{x}_i)) - \lambda_i[F_i(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_p) \\
&\qquad -F_i(x_1, x_2, \ldots, x_{i-1}, \hat{x}_i, x_{i+1}, \ldots, x_p)]\| \\
&\qquad +\lambda_i\|F_i(x_1, x_2, \ldots, x_{i-1}, \hat{x}_i, x_{i+1}, \ldots, x_p)) - F_i(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_p)\| \\
&\qquad +\lambda_i\|G_i(y_{i1}, y_{i2}, \ldots, y_{ip}) - G_i(\hat{y}_{i1}, \hat{y}_{i2}, \ldots, \hat{y}_{ip})\|, \quad i = 1, 2, \ldots, p. \qquad (3.12)
\end{aligned}
$$

Since $F_i : \prod_{k=1}^{p} E_k \longrightarrow E_i$ is (ζ_i, α_i)-relaxed cocoercive with respect to $\hat{g}_i$ in the i-th argument, and β_{ii}-Lipschitz continuous in the i-th argument, respectively, by Lemma 2.1 we get,

$$
\begin{aligned}
&\|A_i(g_i(x_i)) - A_i(g_i(\hat{x}_i)) - \lambda_i[F_i(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_p) \\
&\quad -F_i(x_1, x_2, \ldots, x_{i-1}, \hat{x}_i, x_{i+1}, \ldots, x_p)]\|^q \\
&\le \|A_i(g_i(x_i)) - A_i(g_i(\hat{x}_i))\|^q - q\lambda_i\langle F_i(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_p) \\
&\quad -F_i(x_1, x_2, \ldots, x_{i-1}, \hat{x}_i, x_{i+1}, \ldots, x_p), J_q(A_i(g_i(x_i)) - A_i(g_i(\hat{x}_i)))\rangle + c_q\lambda_i^q \\
&\quad \times\|F_i(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_p) - F_i(x_1, x_2, \ldots, x_{i-1}, \hat{x}_i, x_{i+1}, \ldots, x_p)\|^q \\
&\le ({\delta_i}^q {s_i}^q - q\lambda_i\alpha_i + q\lambda_i\zeta_i{\beta_{ii}}^q + c_q\lambda_i^q{\beta_{ii}}^q)\|x_i - \hat{x}_i\|^q, \quad i = 1, 2, \ldots p. \qquad (3.13)
\end{aligned}
$$

Since $F_i : \prod_{k=1}^{p} E_k \longrightarrow E_i$ is β_{ij}-Lipschitz continuous in the j-th argument for $j = 1, \ldots, i-1, i+1, \ldots, p$, we have,

$$
\begin{aligned}
&\|F_i(x_1, x_2, \ldots, x_{i-1}, \hat{x}_i, x_{i+1}, \ldots, x_p)) - F_i(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_p)\| \\
&\quad \le \|F_i(x_1, x_2, \ldots, x_{i-1}, \hat{x}_i, x_{i+1}, \ldots, x_p)) - F_i(\hat{x}_1, x_2, \ldots, x_{i-1}, \hat{x}_i, x_{i+1}, \ldots, x_p)\| \\
&\qquad +\|F_i(\hat{x}_1, x_2, x_3, \ldots, x_{i-1}, \hat{x}_i, x_{i+1}, \ldots, x_p) \\
&\qquad -F_i(\hat{x}_1, \hat{x}_2, x_3, \ldots, x_{i-1}, \hat{x}_i, x_{i+1}, \ldots, x_p)\| \\
&\qquad +\|F_i(\hat{x}_1, \hat{x}_2, x_3, \ldots, x_{i-1}, \hat{x}_i, x_{i+1}, \ldots, x_p) \\
&\qquad -F_i(\hat{x}_1, \hat{x}_2, \hat{x}_3, \ldots, x_{i-1}, \hat{x}_i, x_{i+1}, \ldots, x_p)\| \\
&\qquad +\cdots \\
&\qquad +\|F_i(\hat{x}_1, \hat{x}_2, \hat{x}_3, \ldots, \hat{x}_{i-2}, x_{i-1}, \hat{x}_i, x_{i+1}, \ldots, x_p) \\
&\qquad -F_i(\hat{x}_1, \hat{x}_2, \hat{x}_3, \ldots, \hat{x}_{i-2}, \hat{x}_{i-1}, \hat{x}_i, x_{i+1}, \ldots, x_p)\| \\
&\qquad +\|F_i(\hat{x}_1, \hat{x}_2, \hat{x}_3, \ldots, \hat{x}_{i-2}, \hat{x}_{i-1}, \hat{x}_i, x_{i+1}, \ldots, x_p) \\
&\qquad -F_i(\hat{x}_1, \hat{x}_2, \hat{x}_3, \ldots, \hat{x}_{i-2}, \hat{x}_{i-1}, \hat{x}_i, \hat{x}_{i+1}, \ldots, x_p)\| \ + \ldots \\
&\qquad \| + F_i(\hat{x}_1, \hat{x}_2, \hat{x}_3, \ldots, \hat{x}_{i-2}, \hat{x}_{i-1}, \hat{x}_i, \hat{x}_{i+1}, \ldots, \hat{x}_{p-1}, x_p) \\
&\qquad -F_i(\hat{x}_1, \hat{x}_2, \ldots, , \hat{x}_{p-1}, \hat{x}_p)\| \\
&\quad \le \beta_{i1}\|x_1 - \hat{x}_1\| + \beta_{i2}\|x_2 - \hat{x}_2\| \\
&\qquad + \cdots + \beta_{i,i-1}\|x_{i-1} - \hat{x}_{i-1}\| + \beta_{i,i+1}\|x_{i+1} - \hat{x}_{i+1}\| + \cdots + \beta_{i,p}\|x_p - \hat{x}_p\| \\
&\quad = \sum_{j=1}^{i-1}\beta_{ij}\|x_j - \hat{x}_j\| + \sum_{j=i+1}^{p}\beta_{ij}\|x_j - \hat{x}_j\|, \qquad i = 1, 2, \ldots p. \qquad (3.14)
\end{aligned}
$$

It follows from the Lipschitz continuity of G_i, the $\tilde{D}$-Lipschitz continuity of T_{ij}, (3.9) that

$$\begin{aligned}
&\|G_i(y_{i1}, y_{i2}, \ldots, y_{ip}) - G_i(\hat{y}_{i1}, \hat{y}_{i2}, \ldots, \hat{y}_{ip})\| \\
&\quad \leq \|G_i(y_{i1}, y_{i2}, \ldots, y_{ip}) - G_i(\hat{y}_{i1}, y_{i2}, \ldots, y_{ip})\| \quad + \|G_i(\hat{y}_{i1}, y_{i2}, \ldots, y_{ip}) \\
&\qquad -G_i(\hat{y}_{i1}, \hat{y}_{i2}, \ldots, y_{ip})\| + \cdots + \|G_i(\hat{y}_{i1}, \hat{y}_{i2}, \ldots, y_{ip}) - G_i(\hat{y}_{i1}, \hat{y}_{i2}, \ldots, \hat{y}_{ip})\| \\
&\quad \leq \sum_{j=1}^{p} \xi_{ij} \|y_{ij} - \hat{y}_{ij}\| \leq \sum_{j=1}^{p} \xi_{ij} \tilde{D}(T_{ij}(x_j), T_{ij}(\hat{x}_j)) \ \leq \sum_{j=1}^{p} \xi_{ij} l_{ij} \|x_j - \hat{x}_j\|, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 1, 2, \ldots, n. \qquad (3.15)
\end{aligned}$$

It follows from (3.10)–(3.15) that for $i = 1, 2, \ldots, p$,

$$\begin{aligned}
\|a_i - b_i\| \leq &\left[\sqrt[q]{1 + c_q s_i^q + q t_i s_i^q - q r_i} + \mu_i \right.\\
&\left. + \frac{\tau_i^{q-1}}{\gamma_i - \lambda_i m_i} \left(\sqrt[q]{{\delta_i}^q {s_i}^q - q\lambda_i \alpha_i + q\lambda_i \zeta_i {\beta_{ii}}^q + c_q \lambda_i^q {\beta_{ii}}^q} + \lambda_i \xi_{ii} l_{ii}\right)\right] \|x_i - \hat{x}_i\| \\
&+ \sum_{j=1}^{i-1} (\beta_{ij} + \xi_{ij} l_{ij}) \frac{\lambda_i \tau_i^{q-1}}{\gamma_i - \lambda_i m_i} \|x_j - \hat{x}_j\| + \sum_{j=i+1}^{p} (\beta_{ij} + \xi_{ij} l_{ij}) \frac{\lambda_i \tau_i^{q-1}}{\gamma_i - \lambda_i m_i} \\
&\qquad\qquad \|x_j - \hat{x}_j\|. \qquad (3.16)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{i=1}^{p} \|a_i - b_i\| \\
&\quad \leq \sum_{i=1}^{p} \left\{ \left[\sqrt[q]{1 + c_q s_i^q + q t_i s_i^q - q r_i} + \mu_i \right.\right.\\
&\qquad \left. + \frac{\tau_i^{q-1}}{\gamma_i - \lambda_i m_i} \left(\sqrt[q]{{\delta_i}^q {s_i}^q - q\lambda_i \alpha_i + q\lambda_i \zeta_i {\beta_{ii}}^q + c_q \lambda_i^q {\beta_{ii}}^q} + \lambda_i \xi_{ii} l_{ii}\right)\right] \|x_i - \hat{x}_i\| \\
&\qquad + \sum_{j=1}^{i-1} (\beta_{ij} + \xi_{ij} l_{ij}) \frac{\lambda_i \tau_i^{q-1}}{\gamma_i - \lambda_i m_i} \|x_j - \hat{x}_j\| \\
&\qquad \left. + \sum_{j=i+1}^{p} (\beta_{ij} + \xi_{ij} l_{ij}) \frac{\lambda_i \tau_i^{q-1}}{\gamma_i - \lambda_i m_i} \|x_j - \hat{x}_j\| \right\} \\
&\quad = \left(\sqrt[q]{1 - q r_1 + q t_1 s_1^q + c_q {s_1}^q} \ + \ \mu_1 \right.\\
&\qquad + \frac{\tau_1^{q-1}}{\gamma_1 - \lambda_1 m_1} \left(\sqrt[q]{{\delta_1}^q {s_1}^q - q\lambda_1 \alpha_1 + q\lambda_1 \zeta_1 {\beta_{11}}^q + c_q \lambda_1^q {\beta_{11}}^q} + \lambda_1 \xi_{11} l_{11}\right) \\
&\qquad \left. + \sum_{j=2}^{p} \frac{\lambda_j \tau_j^{q-1}}{\gamma_j - \lambda_j m_j} (\beta_{j1} + \xi_{j1} l_{j1})\right) \|x_1 - \hat{x}_1\|
\end{aligned}$$

$$
\begin{aligned}
&+\Bigg(\sqrt[q]{1-qr_2+qt_2s_2^q+c_q{s_2}^q}\ +\ \mu_2\\
&+\frac{\tau_2^{q-1}}{\gamma_2-\lambda_2 m_2}\left(\sqrt[q]{{\delta_2}^q{s_2}^q-q\lambda_2\alpha_2+q\lambda_2\zeta_2{\beta_{22}}^q+c_q\lambda_2^q{\beta_{22}}^q}+\lambda_2\xi_{22}l_{22}\right)\\
&+\frac{\lambda_1\tau_1^{q-1}}{\gamma_1-\lambda_1 m_1}(\beta_{12}+\xi_{12}l_{12})+\sum_{j=3}^{p}(\beta_{j2}+\xi_{j2}l_{j2})\frac{\lambda_j\tau_j^{q-1}}{\gamma_j-\lambda_j m_j}\Bigg)\|x_2-\hat{x}_2\|\\
&+\cdots\\
&+\Bigg(\sqrt[q]{1-qr_p+qt_ps_p^q+c_q{s_p}^q}\ +\ \mu_p\\
&+\frac{\tau_p^{q-1}}{\gamma_p-\lambda_p m_p}(\sqrt[q]{{\delta_p}^q{s_p}^q-q\lambda_p\alpha_p+q\lambda_p\zeta_p{\beta_{pp}}^q+c_q\lambda_p^q{\beta_{pp}}^q}+\lambda_p\xi_{pp}l_{pp})\\
&+\sum_{j=1}^{p-1}\frac{\lambda_j\tau_j^{q-1}}{\gamma_j-\lambda_j m_j}(\beta_{jp}+\xi_{jp}l_{jp})\Bigg)\|x_p-\hat{x}_p\|\\
\le{}&\theta\Bigg(\sum_{i=1}^{p}\|x_i-\hat{x}_i\|\Bigg)=\theta\|(x_1,x_2,\ldots,x_p)-(\hat{x}_1,\hat{x}_2,\ldots,\hat{x}_p)\|_*,
\end{aligned}
\tag{3.17}
$$

where

$$
\begin{aligned}
\theta=\max\Bigg\{&\sqrt[q]{1-qr_1+qt_1s_1^q+c_q{s_1}^q}\ +\ \mu_1\\
&+\frac{\tau_1^{q-1}}{\gamma_1-\lambda_1 m_1}\left(\sqrt[q]{{\delta_1}^q{s_1}^q-q\lambda_1\alpha_1+q\lambda_1\zeta_1{\beta_{11}}^q+c_q\lambda_1^q{\beta_{11}}^q}+\lambda_1\xi_{11}l_{11}\right)\\
&+\sum_{j=2}^{p}\frac{\lambda_j\tau_j^{q-1}}{\gamma_j-\lambda_j m_j}(\beta_{j1}+\xi_{j1}l_{j1}),\ \sqrt[q]{1-qr_2+qt_2s_2^q+c_q{s_2}^q}\ +\ \mu_2\\
&+\frac{\tau_2^{q-1}}{\gamma_2-\lambda_2 m_2}\left(\sqrt[q]{{\delta_2}^q{s_2}^q-q\lambda_2\alpha_2+q\lambda_2\zeta_2{\beta_{22}}^q+c_q\lambda_2^q{\beta_{22}}^q}+\lambda_2\xi_{22}l_{22}\right)\\
&+\frac{\lambda_1\tau_1^{q-1}}{\gamma_1-\lambda_1 m_1}(\beta_{12}+\xi_{12}l_{12})+\sum_{j=3}^{p}(\beta_{j2}+\xi_{j2}l_{j2})\frac{\lambda_j\tau_j^{q-1}}{\gamma_j-\lambda_j m_j},\\
&\ldots,\sqrt[q]{1-qr_p+qt_ps_p^q+c_q{s_p}^q}\ +\ \mu_p\\
&+\frac{\tau_p^{q-1}}{\gamma_p-\lambda_p m_p}\left(\sqrt[q]{{\delta_p}^q{s_p}^q-q\lambda_p\alpha_p+q\lambda_p\zeta_p{\beta_{pp}}^q+c_q\lambda_p^q{\beta_{pp}}^q}+\lambda_p\xi_{pp}l_{pp}\right)\\
&+\sum_{j=1}^{p-1}\frac{\lambda_j\tau_j^{q-1}}{\gamma_j-\lambda_j m_j}(\beta_{jp}+\xi_{jp}l_{jp})\Bigg\}.
\end{aligned}
$$

By (3.7), we know that $0<\theta<1$. Hence, from (3.17), we get

$$\begin{aligned}&d((a_1, a_2, \ldots a_p), \Phi_{\lambda_1,\lambda_2,\ldots,\lambda_p}(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_p))\\&= \inf_{(b_1,b_2,\ldots,b_p)\in\Phi_{\lambda_1,\lambda_2,\ldots,\lambda_p}(\hat{x}_1,\hat{x}_2,\ldots,\hat{x}_p))} (\|a_1 - b_1\| + \|a_2 - b_2\| + \cdots + \|a_p - b_p\|)\\&\leq \theta\|(x_1, x_2, \ldots, x_p) - (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_p)\|_*.\end{aligned}$$

Since $(a_1, a_2, \ldots a_p) \in \Phi_{\lambda_1,\lambda_2,\ldots,\lambda_p}(x_1, x_2, \ldots, x_p)$ is arbitrary, we obtain

$$\begin{aligned}&\sup_{(a_1,a_2,\ldots a_p)\in\Phi_{\lambda_1,\lambda_2,\ldots,\lambda_p}(x_1,x_2,\ldots,x_p)} d((a_1, a_2, \ldots a_p), \Phi_{\lambda_1,\lambda_2,\ldots,\lambda_p}(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_p)))\\&\leq \theta\|(x_1, x_2, \ldots, x_p) - (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_p)\|_*.\end{aligned}$$

By using the same argument, we can prove

$$\begin{aligned}&\sup_{(b_1,b_2,\ldots,b_p)\in\Phi_{\lambda_1,\lambda_2,\ldots,\lambda_p}(\hat{x}_1,\hat{x}_2,\ldots,\hat{x}_p)} d(\Phi_{\lambda_1,\lambda_2,\ldots,\lambda_p}(x_1, x_2, \ldots, x_p), (b_1, b_2, \ldots, b_p))\\&\leq \theta\|(x_1, x_2, \ldots, x_p) - (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_p)\|_*.\end{aligned}$$

It follows from the definition of the Hausdorff metric $\tilde{D}$ on $CB(\prod_{j=1}^{p} E_j)$ that for any $(x_1, x_2, \ldots, x_p), (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_p) \in \prod_{j=1}^{p} E_j$

$$\begin{aligned}&\tilde{D}(\Phi_{\lambda_1,\lambda_2,\ldots,\lambda_p}(x_1, x_2, \ldots, x_p), \Phi_{\lambda_1,\lambda_2,\ldots,\lambda_p}(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_p)) \leq \theta\|(x_1, x_2, \ldots, x_p)\\&-(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_p)\|_*.\end{aligned}$$

This proves that $\Phi_{\lambda_1,\lambda_2,\ldots,\lambda_p}$ is a contractive set-valued mapping. Hence, by a fixed point theorem of Nadler [17], $\Phi_{\lambda_1,\lambda_2,\ldots,\lambda_p}$ has a fixed point $(x_1^*, x_2^*, \ldots, x_p^*) \in \prod_{j=1}^{p} E_j$, i.e., $(x_1^*, x_2^*, \ldots, x_p^*) \in \Phi_{\lambda_1,\lambda_2,\ldots,\lambda_p}(x_1^*, x_2^*, \ldots, x_p^*)$. By the definition of $\Phi_{\lambda_1,\lambda_2,\ldots,\lambda_p}$, we know that there exist $y_{ij}^* \in T_{ij}(x_j^*)$ $(i, j = 1, 2, \ldots, p)$ such that for each $i = 1, 2, \ldots, p$,

$$g_i(x_i^*) = R_{M_i(.,x_i^*),\lambda_i,m_i}^{A_i,\eta_i}(A_i(g_i(x_i^*)) - \lambda_i F_i(x_1^*, x_2^*, \ldots, x_p^*) - \lambda_i G_i(y_{i1}^*, y_{i2}^*, \ldots, y_{ip}^*)),$$

It follows from Lemma 3.1 that $(x_1^*, x_2^*, \ldots, x_p^*, y_{11}^*, y_{12}^*, \ldots, y_{1p}^*, y_{21}^*, y_{22}^*, \ldots, y_{2p}^*, \ldots, y_{p1}^*, y_{p2}^*, \ldots, y_{pp}^*)$ is the solution of problem (3.1). This completes the proof. □

Remark 3.1 (i) By Theorem 3.1, it is easy to obtain the existence of solutions for the special cases of problem (3.1). Now we give two examples as follows:

For $i = 1, 2, \ldots, p$, if $M_i(.,.) = M_i(.)$ is an (H_i, η_i)-accretive operators, then $m_i = \mu_i = 0$. By Theorem 3.1, we have

Corollary 3.1 *For $i = 1, 2, \ldots, p$, let $\eta_i : E_i \times E_i \longrightarrow E_i$ be τ_i-Lipshitz continuous, $H_i : E_i \longrightarrow E_i$ be γ_i-strongly η_i-accretive and δ_i-Lipschitz continuous, $g_i : E_i \longrightarrow E_i$ be (t_i, r_i)-relaxed cocoercive and s_i-Lipschitz continuous, $F_i : \prod_{k=1}^{p} E_k \longrightarrow E_i$ be (ζ_i, α_i)-relaxed cocoercive with respect to $\hat{g}_i$ in the i-th argument, β_{ij}-Lipschitz continuous in the j-th argument for $j = 1, \ldots, i-1, i, i+1, \ldots, p$, where $\hat{g}_i : E_i \longrightarrow E_i$ is defined by $\hat{g}_i(x_i) = A_i \circ g_i(x_i) = A_i(g_i(x_i)), \forall x_i \in E_i$, and $G_i : \prod_{k=1}^{p} E_k \longrightarrow E_i$ be ξ_{ij}-Lipschitz continuous in the j-th argument for $j = 1, 2, \ldots, p$, $M_i : E_i \longrightarrow 2^{E_i}$ be (H_i, η_i)-accretive, and the set-valued mappings $T_{1i} : E_i \longrightarrow CB(E_i)$, $T_{2i} : E_i \longrightarrow CB(E_i), \ldots, T_{pi} : E_i \longrightarrow CB(E_i)$ be l_{1i}-$\tilde{D}$-Lipschitz continuous, l_{2i}-$\tilde{D}$-Lipschitz,..., l_{pi}-$\tilde{D}$-Lipschitz continuous, respectively. If there exist constants $\lambda_i > 0$ $(i = 1, 2, \ldots, p)$ such that*

$$\begin{cases} \sqrt[q]{1 - qr_1 + qt_1 s_1^q + c_q {s_1}^q} \quad + \frac{\tau_1^{q-1}}{\gamma_1}\left(\sqrt[q]{{\delta_1}^q {s_1}^q - q\lambda_1\alpha_1 + q\lambda_1\zeta_1{\beta_{11}}^q + c_q\lambda_1^q{\beta_{11}}^q}\right. \\ \qquad \left. + \lambda_1\xi_{11}l_{11}\right) + \sum_{j=2}^{p}\frac{\lambda_j\tau_j^{q-1}}{\gamma_j}(\beta_{j1} + \xi_{j1}l_{j1}) < 1, \\ \sqrt[q]{1 - qr_2 + qt_2 s_2^q + c_q {s_2}^q} \quad + \frac{\tau_2^{q-1}}{\gamma_2}\left(\sqrt[q]{{\delta_2}^q {s_2}^q - q\lambda_2\alpha_2 + q\lambda_2\zeta_2{\beta_{22}}^q + c_q\lambda_2^q{\beta_{22}}^q}\right. \\ \qquad \left. + \lambda_2\xi_{22}l_{22}\right) \\ + \frac{\lambda_1\tau_1^{q-1}}{\gamma_1}(\beta_{12} + \xi_{12}l_{12}) + \sum_{j=3}^{p}(\beta_{j2} + \xi_{j2}l_{j2})\frac{\lambda_j\tau_j^{q-1}}{\gamma_j} < 1, \\ \ldots \\ \sqrt[q]{1 - qr_p + qt_p s_p^q + c_q {s_p}^q} \quad + \frac{\tau_p^{q-1}}{\gamma_p}\left(\sqrt[q]{{\delta_p}^q {s_p}^q - q\lambda_p\alpha_p + q\lambda_p\zeta_p{\beta_{pp}}^q + c_q\lambda_p^q{\beta_{pp}}^q}\right. \\ \qquad \left. + \lambda_p\xi_{pp}l_{pp}\right) \\ + \sum_{j=1}^{p-1}\frac{\lambda_j\tau_j^{q-1}}{\gamma_j}(\beta_{jp} + \xi_{jp}l_{jp}) < 1. \end{cases}$$

Then problem (3.2) admits a solution $(x_1, x_2, \ldots, x_p, y_{11}, y_{12}, \ldots, y_{1p}, y_{21}, y_{22}, \ldots, y_{2p}, \ldots, y_{p1}, y_{p2}, \ldots, y_{pp})$.

For $i, j = 1, 2, \ldots, p$, if $M_i(.,.) = M_i(.)$ is an (A_i, η_i, m_i)-accretive operators, $T_{ij} \equiv I_j$, then $l_{ij} = 1$ and $\mu_i = 0$. By Theorem 3.1, we have:

Corollary 3.2 *For $i = 1, 2, \ldots, p$, let $\eta_i : E_i \times E_i \longrightarrow E_i$ be τ_i-Lipshitz continuous, $A_i : E_i \longrightarrow E_i$ be γ_i-strongly η_i-accretive and δ_i-Lipschitz continuous, $g_i : E_i \longrightarrow E_i$ be (t_i, r_i)-relaxed cocoercive and s_i-Lipschitz continuous, $F_i : \prod_{k=1}^{p} E_k \longrightarrow E_i$ be (ζ_i, α_i)-relaxed cocoercive with respect to $\hat{g}_i$ in the i-th argument, β_{ij}-Lipschitz continuous in the j-th argument for $j = 1, \ldots, i-1, i, i+1, \ldots, p$, where $\hat{g}_i : E_i \longrightarrow E_i$ is defined by $\hat{g}_i(x_i) = A_i \circ g_i(x_i) = A_i(g_i(x_i)), \forall x_i \in E_i$, and $G_i : \prod_{k=1}^{p} E_k \longrightarrow E_i$ be ξ_{ij}-Lipschitz continuous in the j-th argument for $j = 1, 2, \ldots, p$, $M_i : E_i \times E_i \longrightarrow 2^{E_i}$ be (A_i, η_i, m_i)-accretive in the first argument. If there exist constants $\lambda_i > 0$ $(i = 1, 2, \ldots, p)$ such that*

$$\begin{cases} \sqrt[q]{1 - qr_1 + qt_1 s_1^q + c_q {s_1}^q} \; + \frac{\tau_1^{q-1}}{\gamma_1 - \lambda_1 m_1}\left(\sqrt[q]{{\delta_1}^q {s_1}^q - q\lambda_1\alpha_1 + q\lambda_1\zeta_1{\beta_{11}}^q + c_q\lambda_1^q{\beta_{11}}^q}\right. \\ \qquad \left. + \lambda_1\xi_{11}\right) \\ + \sum_{j=2}^{p}\frac{\lambda_j\tau_j^{q-1}}{\gamma_j - \lambda_j m_j}(\beta_{j1} + \xi_{j1}) < 1, \sqrt[q]{1 - qr_2 + qt_2 s_2^q + c_q {s_2}^q} \\ + \frac{\tau_2^{q-1}}{\gamma_2 - \lambda_2 m_2}\left(\sqrt[q]{{\delta_2}^q {s_2}^q - q\lambda_2\alpha_2 + q\lambda_2\zeta_2{\beta_{22}}^q + c_q\lambda_2^q{\beta_{22}}^q} + \lambda_2\xi_{22}\right) \\ + \frac{\lambda_1\tau_1^{q-1}}{\gamma_1 - \lambda_1 m_1}(\beta_{12} + \xi_{12}) + \sum_{j=3}^{p}(\beta_{j2} + \xi_{j2})\frac{\lambda_j\tau_j^{q-1}}{\gamma_j - \lambda_j m_j} < 1, \\ \ldots \\ \sqrt[q]{1 - qr_p + qt_p s_p^q + c_q {s_p}^q} + \frac{\tau_p^{q-1}}{\gamma_p - \lambda_p m_p}\left(\sqrt[q]{{\delta_p}^q {s_p}^q - q\lambda_p\alpha_p + q\lambda_p\zeta_p{\beta_{pp}}^q + c_q\lambda_p^q{\beta_{pp}}^q}\right. \\ \qquad \left. + \lambda_p\xi_{pp}\right) + \sum_{j=1}^{p-1}\frac{\lambda_j\tau_j^{q-1}}{\gamma_j - \lambda_j m_j}(\beta_{jp} + \xi_{jp}) < 1. \end{cases}$$

Then problem (3.3) admits a solution $(x_1, x_2, \dots, x_p, y_{11}, y_{12}, \dots, y_{1p}, y_{21}, y_{22}, \dots, y_{2p}, \dots, y_{p1}, y_{p2}, \dots, y_{pp})$.

Remark 3.2 For $i = 1, 2, \dots, p$, if $t_i = \zeta_i = 0$ in Theorem 3.1, then $g_i : E_i \longrightarrow E_i$ becomes r_i-strongly accretive, $F_i : \prod_{k=1}^{p} E_k \longrightarrow E_i$ becomes α_i-strongly accretive with respect to $\hat{g}_i$ in the i-th argument, and the hypothesis (3.7) can be replaced by the following formula:

$$\begin{cases} \sqrt[q]{1-qr_1+c_q s_1{}^q} + \mu_1 + \frac{\tau_1^{q-1}}{\gamma_1-\lambda_1 m_1}\left(\sqrt[q]{\delta_1{}^q s_1{}^q - q\lambda_1\alpha_1 + c_q\lambda_1^q\beta_{11}{}^q} + \lambda_1\xi_{11}l_{11}\right) \\ + \sum_{j=2}^{p}\frac{\lambda_j\tau_j^{q-1}}{\gamma_j-\lambda_j m_j}(\beta_{j1}+\xi_{j1}l_{j1}) < 1,\ \sqrt[q]{1-qr_2+c_q s_2{}^q} + \mu_2 \\ + \frac{\tau_2^{q-1}}{\gamma_2-\lambda_2 m_2}\left(\sqrt[q]{\delta_2{}^q s_2{}^q - q\lambda_2\alpha_2 + c_q\lambda_2^q\beta_{22}{}^q} + \lambda_2\xi_{22}l_{22}\right) \\ + \frac{\lambda_1\tau_1^{q-1}}{\gamma_1-\lambda_1 m_1}(\beta_{12}+\xi_{12}l_{12}) + \sum_{j=3}^{p}(\beta_{j2}+\xi_{j2}l_{j2})\frac{\lambda_j\tau_j^{q-1}}{\gamma_j-\lambda_j m_j} < 1, \\ \dots \\ \sqrt[q]{1-qr_p+c_q s_p{}^q} + \mu_p + \frac{\tau_p^{q-1}}{\gamma_p-\lambda_p m_p}\left(\sqrt[q]{\delta_p{}^q s_p{}^q - q\lambda_p\alpha_p + c_q\lambda_p^q\beta_{pp}{}^q} + \lambda_p\xi_{pp}l_{pp}\right) \\ + \sum_{j=1}^{p-1}\frac{\lambda_j\tau_j^{q-1}}{\gamma_j-\lambda_j m_j}(\beta_{jp}+\xi_{jp}l_{jp}) < 1. \end{cases}$$

Remark 3.3 The novel proof approach for the existence of solutions used in Theorem 3.1 is different from those iterative methods used in [8–13] and is suitable for the proofs of existence results for all kinds of classes of system of set-valued variational inclusions.

We introduce the following p-step iterative algorithm for solving problem (3.1) as following:

Algorithm 3.1 For any given $x_i^0 \in E_i$ $(i = 1, 2, \dots, p)$, we can compute the sequences $x_i^n, y_{1i}^n, y_{2i}^n, \dots, y_{pi}^n$ $(i = 1, 2, \dots, p)$ by the following p-step iterative schemes such that for each $i = 1, 2, \dots, p$,

$$\begin{aligned} x_i^{n+1} &= x_i^n - g_i(x_i^n) + R^{A_i,\eta_i}_{M_i(\cdot,x_i^n),\lambda_i,m_i}(A_i(g_i(x_i^n)) - \lambda_i F_i(x_1^n, x_2^n, \dots, x_p^n) \\ &\quad -\lambda_i G_i(y_{i1}^n, y_{i2}^n, \dots, y_{ip}^n)). \\ &\quad y_{1i}^n \in T_{1i}(x_i^n),\ \|y_{1i}^n - y_{1i}^{n-1}\| \le \left(1+\frac{1}{n}\right)\tilde{D}(T_{1i}(x_i^n), T_{1i}(x_i^{n-1})), \\ &\quad y_{2i}^n \in T_{2i}(x_i^n),\ \|y_{2i}^n - y_{2i}^{n-1}\| \le \left(1+\frac{1}{n}\right)\tilde{D}(T_{2i}(x_i^n), T_{2i}(x_i^{n-1})), \\ &\quad \dots \\ &\quad y_{pi}^n \in T_{pi}(x_i^n),\ \|y_{pi}^n - y_{pi}^{n-1}\| \le \left(1+\frac{1}{n}\right)\tilde{D}(T_{pi}(x_i^n), T_{pi}(x_i^{n-1})). \end{aligned}$$

for all $n = 0, 1, 2, \dots$. By similar argument with that in [13], we can prove the existence of a solution of problem (3.1) and the convergence of Algorithm 3.1 as follows:

Theorem 3.2 *For $i = 1, 2, \dots, p$, let $\eta_i : E_i \times E_i \longrightarrow E_i$ be τ_i-Lipshitz continuous, $A_i : E_i \longrightarrow E_i$ be γ_i-strongly η_i-accretive and δ_i-Lipschitz continuous, $g_i : E_i \longrightarrow E_i$ be (t_i, r_i)-relaxed cocoercive and s_i-Lipschitz continuous, $F_i : \prod_{k=1}^{p} E_k \longrightarrow E_i$ be (ζ_i, α_i)-relaxed cocoercive with respect to $\hat{g}_i$ in the i-th argument, β_{ij}-Lipschitz continuous in the*

j-th argument for $j = 1, \ldots, i-1, i, i+1, \ldots, p$, where $\hat{g}_i : E_i \longrightarrow E_i$ is defined by $\hat{g}_i(x_i) = A_i \circ g_i(x_i) = A_i(g_i(x_i)), \forall x_i \in E_i$, and $G_i : \prod_{k=1}^{p} E_k \longrightarrow E_i$ be ξ_{ij}-Lipschitz continuous in the j-th argument for $j = 1, 2, \ldots, p$, $M_i : E_i \times E_i \longrightarrow 2^{E_i}$ be (A_i, η_i, m_i)-accretive in the first argument, and the set-valued mappings $T_{1i} : E_i \longrightarrow CB(E_i)$, $T_{2i} : E_i \longrightarrow CB(E_i)$,...,$T_{pi} : E_i \longrightarrow CB(E_i)$ be l_{1i}-$\tilde{D}$-Lipschitz continuous, l_{2i}-$\tilde{D}$-Lipschitz,..., l_{pi}-$\tilde{D}$-Lipschitz continuous, respectively. In addition, if

$$\| R^{A_i,\eta_i}_{M_i(.,x_i),\lambda_i,m_i}(z_i) - R^{A_i,\eta_i}_{M_i(.,\hat{x}_i),\lambda_i,m_i}(z_i) \| \le \mu_i \| x_i - \hat{x}_i \|, \quad \forall x_i, \hat{x}_i, z_i \in E_i,$$

for all $i = 1, 2, \ldots, p$. If there exist constants λ_i such that $0 < \lambda_i < \frac{\gamma_i}{m_i}$ $(i = 1, 2, \ldots, p)$ and the hypothesis (3.7) holds. Then problem (3.1) admits a solution $(x_1, x_2, \ldots, x_p, y_{11}, y_{12}, \ldots, y_{1p}, y_{21}, y_{22}, \ldots, y_{2p}, \ldots, y_{p1}, y_{p2}, \ldots, y_{pp})$ and sequences $x_1^n, x_2^n, \ldots, x_p^n$, $y_{11}^n, y_{12}^n, \ldots, y_{1p}^n, y_{21}^n, y_{21}^n, \ldots, y_{2p}^n, \ldots, y_{p1}^n, y_{p2}^n, \ldots, y_{pp}^n$ converge to $x_1, x_2, \ldots, x_p, y_{11}, y_{12}, \ldots, y_{1p}, y_{21}, y_{22}, \ldots, y_{2p}, \ldots, y_{p1}, y_{p2}, \ldots, y_{pp}$, respectively, where x_i^n, y_{1i}^n, $y_{2i}^n, \ldots, y_{pi}^n$ $(i = 1, 2, \ldots, p)$ are the sequences generated by Algorithm 3.1.

Remark 3.4 Both Algorithm 3.1 and Theorem 3.2 are the corrected versions of the main results in [13].

Remark 3.5 Theorem 3.1 and 3.2 unify, generalize and improve those corresponding results in [1–10] and the references therein.

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